

Spin-Statistics Connection for Relativistic Quantum Mechanics

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The spin-statistics connection has been proved for nonrelativistic quantum mechanics (Jabs, A., 2010: Found. Phys., **40**, 776-792). The proof is extended here to the relativistic regime using the parametrized Dirac equation. A causality condition is not required.

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I. INTRODUCTION

The spin-statistics connection is regarded as one of the most important results in theoretical physics [1–4]. The standard proof in Quantum Field Theory requires relativistic physics, yet it has been argued that spin is intrinsically a nonrelativistic phenomenon [5] since it characterizes the representations of $SO(3)$. On the other hand the electron gyromagnetic ratio $g = 2$ is a consequence of the relativistic wave equation of Dirac, but must be specified in the nonrelativistic wave equation of Pauli [6].

There is an elementary proof in nonrelativistic quantum mechanics for the spin-statistics connection [7]. The objective of this investigation is an elementary extension of the proof to the relativistic regime, using the parametrized Dirac equation attributed to Feynman and others [8]. Jabs [7] briefly outlines an alternative relativistic approach involving helicity, but the helicity operator is only Lorentz-invariant for massless particles. The argument here follows naturally from the nonrelativistic proof, and involves the relativistic rotation operator which is Lorentz-covariant regardless of mass. It is emphasized that the proof is that of a *spin*-statistics theorem, since the states are in a spin-1/2 representation and its dual.

The contents of this article are as follows. Section II includes a statement of the single-particle parametrized Dirac equation with an outline of its utility, the forms of the free plane wave solutions, the splitting of positive-energy plane waves into positive-energy waves propagating forward and backward in coordinate time, and the equation for many particles. The eigenstates of the rotation operator are defined in Section III, where it is proved that any free state at any parameter value evolves from a uniquely determined free rotation eigenstate prepared at an earlier (algebraically smaller) parameter value. The proof of the nonrelativistic spin-statistics connection by Jabs [7] is extended in Section IV to the relativistic regime at the preparation parameter value. The multiple-particle analysis of Jabs is repeated, for the case of two spin-1/2 particles prepared in rotation eigenstates, without a significant loss of generality but with a significant gain in simplicity. The free two-particle influence function for other parameter values, while mixing the rotation eigenstates for each particle, preserves the antisymmetric or fermionic form of the two-particle state. The argument adapts to spin-1 using the parametrization of the Duffin-Kemmer-Petiau equation [9–11], leading to symmetric or bosonic forms. The result is discussed in the concluding

Section V, in particular the lack of need for a causality condition.

II. THE PARAMETRIZED DIRAC EQUATION

A. Covariant Formulation

The wavefunction for a single spin-1/2 particle is a four-spinor $\psi(x, \tau)$. The event x is in \mathbb{R}^4 , the parameter τ is an independent variable in \mathbb{R} . The event x is also denoted by x^μ having indices $\mu = 0, 1, 2, 3$, with $x^0 = ct$ where c is the speed of light and t is coordinate time. The Lorentz metric $g^{\mu\nu}$ on \mathbb{R}^4 has signature $(-+++)$. The position \mathbf{x} is denoted by x^j having indices $j = 1, 2, 3$, and so $x = (ct, \mathbf{x})$.

The parametrized Dirac equation for ψ is

$$\frac{\hbar}{ic} \frac{\partial}{\partial \tau} \psi + \gamma^\mu \left(\frac{\hbar}{i} \frac{\partial}{\partial x^\mu} - \frac{e}{c} A_\mu \right) \psi = 0 \quad (1)$$

where \hbar is the reduced Planck's constant and e is the charge of the particle, while the γ^μ are the four Dirac matrices. The Maxwell electromagnetic potential $A^\mu(x)$ is independent of the parameter τ . The parametric role of τ is clear for a wave packet: at large negative τ the packet may be localized in one region of space-time, and at large positive τ in another region. That is, τ parametrizes the evolution of the space-time moments of the packet. The covariance of the theory with respect to the homogeneous Lorentz transformation $(x^\mu)' = \Omega^\mu{}_\nu x^\nu$, represented by $\psi'(x, \tau) = S(\Omega)\psi(\Omega^{-1}x, \tau)$, follows for $S(\Omega)$ generated in the standard way [6]. There is no mass constant in (1), but masses are introduced through boundary conditions as $\tau \rightarrow \pm\infty$. Feynman's development of Quantum Electrodynamics using (1) is reviewed by Garcia Alvarez and Gaioli [8]. A simple consequence of (1) is the identity

$$\frac{\partial}{\partial \tau} \bar{\psi} \psi + \frac{\partial}{\partial x^\mu} j^\mu = 0 \quad (2)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, and the τ -dependent current is $j^\mu = c \bar{\psi} \gamma^\mu \psi$. The indefiniteness of the conserved, invariant bilinear form $\bar{\psi} \psi$ has impeded the development of the parametrized Dirac formalism as a relativistic extension of quantum mechanics [12, 13]. In the case of a wavefunction having 'sharp mass m_e ' as in $\psi(x, \tau) = \exp(im_e \tau) \psi(x, 0)$, and low kinetic energy $|\mathbf{p}|^2 \ll m_e^2$ where \mathbf{p} is a three-momentum representative of the wavefunction, the Dirac four-spinor $\exp(im_e x^0) \psi(x, 0)$ may be projected onto a 'large upper' and a 'small lower' Pauli two-spinor, both varying slowly in coordinate time x^0 . The 'large upper' component,

denoted φ in [6], satisfies the Pauli or spin-1/2 Schrödinger equation [6, Ch. 1] which conserves the Hermitian form $\varphi^\dagger\varphi$. The situation here is as for quantum mechanics (QM) and Quantum Field Theory (QFT): at low energy the parametrized Dirac equation has a Born interpretation, but at high energy it yields only transition amplitudes.

The discrete symmetries for (1) are given in [14][15]. The electromagnetic potential A^μ is calculated semiclassically in [14], that is, the source for the potential is a Møller current [6, 16]. An expansion of (1) in powers of the fine structure constant $\alpha = e^2/4\pi\hbar c$ yields the Mott scattering cross-section at first order. Combined with a partial summation that is justified for weak scattering, expansion to higher orders yields the Uehling potential, the self-energy and self-mass of fermion lines, the anomalous magnetic moment of the electron, the Lamb shift and the axial anomaly of Quantum Electrodynamics (QED) [14]. The standard cross sections for pair interactions, such as electron-positron annihilation, may be derived from the multiple-particle parametrized Dirac equation (see Section IID) without recourse to hole theory. In brief, two particles are interacting. The real electron scatters off a real photon into a virtual electron, and similarly for the positron. Summing the products of the two amplitudes over the virtual energy-momenta yields the internal fermion line in the standard diagram for pair annihilation into two photons. The multiple-particle parametrized Dirac equation also yields the Bethe-Salpeter equation for bound states without further conjecture [14]. Semiclassical electrodynamics in the presence of an isotropic zero-point electromagnetic far field having energy density $\hbar\omega/2$ at frequency ω yields the blackbody spectrum, the van der Waals forces, the Casimir effect, the Einstein ‘A’ coefficient for spontaneous emission and the photoelectric effect [17–21]. It is widely held that a classical theory of radiation cannot explain the sub-Poisson statistics and antibunching routinely observed [22] in photon counting at very low levels of illumination, but see [23, Ch. 13]. In summary, the parametrized Dirac equation for spin-1/2 particles interacting through a semiclassical potential has extensive utility.

The units are now chosen such that $c = \hbar = 1$. The summation convention is assumed for Greek indices such as $\mu = 0, 1, 2, 3$. The covariant and contravariant indices μ, ν, \dots will be omitted where convenient, as in $x = x^\mu$, $p = p^\mu$, $p \cdot x = p^\mu x_\mu$.

B. Plane Waves

Plane wave solutions of (1) for a vanishing potential $A^\mu = 0$ have the forms

$$\psi_\pm^{(r)}(x, \tau) = w_\pm^{(r)}(p) \exp[i(p \cdot x \pm \varphi_p m_p \tau)] \quad (3)$$

where $w_\pm^{(r)}(p)$ are covariant Dirac four-spinors for $r = 1, 2$, $m_p = \sqrt{-p \cdot p} > 0$ is the positive mass for a subluminal state, and $\varphi_p = \text{sign}(p^0) = p^0/E_p$ where p^0 is the energy with magnitude $E_p = |p^0|$. The phases of the complex exponentials are $p \cdot x \pm \varphi_p m_p \tau$, hence the rates of change of the coordinate time $t = x^0$ with respect to the parameter τ at constant phase and position \mathbf{x} are

$$\frac{dt}{d\tau} = \pm m_p/E_p \quad (4)$$

regardless of the value of φ_p . That is, $\psi_+^{(r)}$ and $\psi_-^{(r)}$ propagate forward and backward in time respectively, regardless of the sign of the energy p^0 . The four-spinors $w_\pm^{(r)}(p)$ are eigenvectors of the energy-momentum projections $\Lambda_\pm(p) = (1/2m_p)(m_p \mp \varphi_p \not{p})$ which satisfy $\Lambda_+ + \Lambda_- = 1$, $\Lambda_\pm \Lambda_\pm = \Lambda_\pm$ and $\Lambda_\pm \Lambda_\mp = 0$. That is, $w_+^{(r)} = \Lambda_+ u^{(r)}$ and $w_-^{(r)} = \Lambda_- v^{(r)}$ for some four-spinors $u^{(r)}$ and $v^{(r)}$, where $r = 1, 2$.

C. Splitting

Suppose that at $\tau = 0$ the state is of positive energy and has the plane wavefunction

$$\psi(x, 0) = w \exp[ip \cdot x], \quad (5)$$

where $p^0 > 0$, $w = (\mathbf{w}_+ \mathbf{a}_+ + \mathbf{w}_- \mathbf{a}_-)$, $\mathbf{w}_\pm = (w_\pm^{(1)}, w_\pm^{(2)})$, and the coefficients \mathbf{a}_\pm are any 2×1 complex matrices. Evolving the state with the free influence function Γ_+^0 as constructed in [14] yields, for $\tau > 0$,

$$\begin{aligned} \psi(x, \tau) &= \frac{1}{i} \int d^4 x' \Gamma_+^0(x - x', \tau) \psi(x', 0) \\ &= \mathbf{w}_+ \mathbf{a}_+ \exp[i(p \cdot x + \varphi_p m_p \tau)]. \end{aligned} \quad (6)$$

For $\tau > 0$ the free influence function Γ_+^0 projects the prepared positive-energy wavefunction $\psi(x, 0)$ onto the free positive-energy wavefunctions that propagate forward in time as τ increases. If the prepared state has negative energy then Γ_+^0 projects onto the free negative-energy wavefunctions that propagate backward in time. The \mathcal{TCP} -conjugate of a wavefunction for a particle is by definition the wavefunction for the oppositely-charged particle or

‘antiparticle’ having energy of opposite sign and propagating in the opposite sense in time. There are antiparticles of positive and negative energy, propagating in either sense in time [14].

D. Many Particles

The parametrized Dirac equation (1) has a natural extension for many particles in the space spanned by tensor products of single-particle wavefunctions. A two-particle wavefunction $\Psi(x, y, \tau)$ satisfies [14]

$$\frac{1}{i} \frac{\partial}{\partial \tau} \Psi + \not{x} \otimes I_4 \Psi + I_4 \otimes \not{y} \Psi = 0 \quad (7)$$

where $\pi^\mu(x) = (1/i)\partial/\partial x_\mu - e_1 A^\mu(x)$ and $\pi^\mu(y) = (1/i)\partial/\partial y_\mu - e_2 A^\mu(y)$. The free influence function $\Gamma_{++}^0(x - x', y - y', \tau - \tau')$ is [14]

$$\Gamma_{++}^0(x - x', y - y', \tau - \tau') = \frac{1}{i} \Gamma_+^0(x - x', \tau - \tau') \otimes \Gamma_+^0(y - y', \tau - \tau'). \quad (8)$$

If there is a nonvanishing external field A^μ , and if $\Gamma_+(x, \tau; x', \tau')$ is the single-particle influence function constructed using $\Gamma_+^0(x - x', \tau - \tau')$ as in [6, 14], then the influence function for (7) is

$$\Gamma_{++}(x, y, \tau; x', y', \tau') = \frac{1}{i} \Gamma_+(x, \tau; x', \tau') \otimes \Gamma_+(y, \tau; y', \tau'). \quad (9)$$

III. ROTATION EIGENSTATES

For any pure spacelike four-vector $s = (0, \mathbf{s})$, with $s \cdot s = \mathbf{s} \cdot \mathbf{s} = +1$, there is the rotation operator $R(s) = (1/2)\gamma^0 \not{s} \gamma^5$. The eigenvalues of $R(s)$ are $l = \pm 1/2$. The associated projections are $P(s) = R(s) + 1/2$ and $Q(s) = 1 - P(s)$. The rotation operator $R(s)$ commutes with the energy-momentum projections $\Lambda_\pm(p)$ if and only if the rotation axis and the 3-momentum are parallel. That is, $[R(0, \mathbf{s}), \Lambda_\pm(p^0, \mathbf{p})] = 0 \iff \mathbf{p} \times \mathbf{s} = \mathbf{0}$. Owing to the two-fold degeneracy of both eigenvalues of $R(s)$ and also of both eigenvalues of say $\Lambda_+(p)$, an eigenvector of either operator is not necessarily an eigenvector of the other even if \mathbf{s} and \mathbf{p} are parallel.

Consider the case in which $\mathbf{p} \times \mathbf{s} \neq \mathbf{0}$ and the wavefunction $\psi(x, \tau)$ defined in (6) for $\tau > 0$ is not an eigenvector of $R(s)$. It is shown in the Appendix that for any \mathbf{a}_+ as in

(6) there is a unique solution for \mathbf{a}_- such that the four-spinor $w = \mathbf{w}_+\mathbf{a}_+ + \mathbf{w}_-\mathbf{a}_-$ satisfies $R(s)w = +(1/2)w$. That is, for any positive-energy, forward-propagating plane wavefunction $\psi(x, \tau)$, where $\tau > 0$, there is a unique, positive-energy plane wavefunction $\psi(x, 0)$ such that $R(s)\psi(x, 0) = (1/2)\psi(x, 0)$ and $\psi(x, 0)$ evolves into $\psi(x, \tau)$ for $\tau > 0$.

If $\mathbf{p} \times \mathbf{s} = \mathbf{0}$ there is in general no solution for \mathbf{a}_- such that the wavefunction $\psi(x, 0)$ is an eigenvector of $R(s)$. However, given a timelike energy-momentum p^μ , another pure spacelike rotation axis s^μ may be chosen in a non-rest frame such that $\mathbf{p} \times \mathbf{s} \neq \mathbf{0}$.

Finally consider, for $\tau > 0$, a positive-energy, forward-propagating, separable two-particle plane wavefunction $\Psi(x, y, \tau)$ with energy-momenta p and q respectively, where $p^0 > 0$ and $q^0 > 0$. Owing to the separability of the two-particle influence function (8) there is a positive-energy plane wavefunction $\Psi(x, y, 0)$ such that, for some pure spacelike rotation axis s^μ , $R(s) \otimes R(s) \Psi(x, y, 0) = (1/4)\Psi(x, y, 0)$ and $\Psi(x, y, 0)$ evolves into $\Psi(x, y, \tau)$ for $\tau > 0$. Note especially that the same axis s^μ is specified for both particles. The construction may be extended to any number of particles, with only a single axis s^μ specified. As discussed in the Appendix, the case $\mathbf{p} \times \mathbf{s} = \mathbf{0}$ for any subset of many particles can be avoided by an appropriate choice of the single axis s^μ .

It need not be assumed that mass is an intrinsic property of a particle. If it is so assumed then indistinguishable particles all have the same mass m_e , and $m_e^2 = -p \cdot p$ for all plane wavefunctions with energy-momentum p^μ . Free wavefunctions including those prepared as rotation eigenvectors at $\tau = 0$ then satisfy the free Klein-Gordon equation [6]. It may be recalled that free solutions of the sharp-mass Dirac equation are also free solutions of the Klein-Gordon equation, but not vice-versa.

IV. SPIN AND STATISTICS

A. At $\tau = 0$

1. same spin

Consider many free, indistinguishable spin-1/2 particles. Assume further that the wavefunctions are all rotation eigenvectors for the same rotation axis s , all with the same eigenvalue $l = -1/2$ or $l = +1/2$. Note that an axis $s = (0, \mathbf{s})$ has been specified but, as pointed out by Jabs [7], not a spatial frame. Thus the referencing of each single-particle

wavefunction to a common frame involves a spatial rotation through an angle χ , which can be restricted to the range $[0, 2\pi]$. It follows [6] that the rotation augments the phase of that spin-1/2 wavefunction by $l\chi$. So in any common frame each single-particle wavefunction is defined only to within a factor of $\exp(il\chi)$. This is precisely the same as the nonrelativistic situation investigated by Jabs [7], who points out that indistinguishability requires the exchange of particles to involve also an exchange of frame-dependent angles. That exchange can be made by, say, a counterclockwise rotation. Demanding further that the exchange be made in an homotopically consistent way has significant consequences. For example, consider the separable two-particle wavefunction

$$\Psi = \psi(x, \tau, l, \chi) \otimes \phi(y, \tau, l, \lambda). \quad (10)$$

Jabs [7] emphasizes that the frame angles χ and λ are parameters rather than observables. First, exchange the wavefunctions ψ and ϕ but not the phases χ and λ , and then add the result to Ψ to obtain

$$\Psi_{S'} = \psi(x, \tau, l, \chi) \otimes \phi(y, \tau, l, \lambda) + \phi(x, \tau, l, \chi) \otimes \psi(y, \tau, l, \lambda). \quad (11)$$

Such simple addition is stipulated for the wavefunctions of indistinguishable particles, whenever a transition to either state is possible. Second, exchange the frame angles in the rightmost summand in (11) by, say, counterclockwise rotation. If, say, $\chi < \lambda$, run χ through $\lambda - \chi$ to obtain a rotation factor $\exp[i(\lambda - \chi)]$, and run λ through $2\pi - \lambda + \chi$ to obtain a factor $\exp[i(2\pi - \lambda + \chi)]$. Then (11) becomes

$$\Psi_S = \psi(x, \tau, l, \chi) \otimes \phi(y, \tau, l, \lambda) - \phi(x, \tau, l, \lambda) \otimes \psi(y, \tau, l, \chi). \quad (12)$$

The frame-angle dependence of the wavefunctions ψ and ϕ are of the form $\psi(x, \tau, l, \chi) = \psi(x, \tau, l) \exp(il\chi)$ and $\phi(y, \tau, l, \lambda) = \phi(y, \tau, l) \exp(i\lambda l)$, hence referring the wavefunctions to their original spin frames yields the common factor $\exp[-il(\chi + \lambda)]$ for the two summands in (12). The common factor and the frame-dependent angles χ and λ may therefore be ignored in the calculation of observable quantities, such as cross-sections, using (12). In conclusion, the wavefunction for two indistinguishable particles in the same rotation eigenstate is antisymmetric with respect to particle exchange [7].

2. different rotation

Consider two otherwise indistinguishable particles in eigenstates of opposite rotation, that is, opposite values of l . The separable two-particle wavefunction is

$$\Psi = \psi(x, \tau, l, \chi) \otimes \phi(y, \tau, n, \lambda) \quad (13)$$

where $|l| = |n| = 1/2$ and $l = -n$. Exchanging wavefunctions but not angles and then adding yields

$$\Psi_{S'} = \psi(x, \tau, l, \chi) \otimes \phi(y, \tau, n, \lambda) + \phi(x, \tau, n, \chi) \otimes \psi(y, \tau, l, \lambda). \quad (14)$$

An homotopically-consistent exchange of angles in the rightmost summand in (14) assuming, say, counterclockwise rotation and $\chi < \lambda$ yields

$$\Psi_S = \psi(x, \tau, l, \chi) \otimes \phi(y, \tau, n, \lambda) - \kappa \phi(x, \tau, n, \lambda) \otimes \psi(y, \tau, l, \chi), \quad (15)$$

where $\kappa = \exp[i(l - n)(\lambda - \chi)]$. The wavefunction Ψ_S in (15), which of course is valid also for the case $l = n$, is not antisymmetric if $l \neq n$ and so the state is therefore inconsistent with the exclusion principle. The case $\chi = \lambda$ is ignored for having vanishing measure [7]. Following Feynman [24], Jabs argues that physical significance resides not in wavefunctions, but rather in transition amplitudes f which must be calculated as

$$f = \int d^4x \int d^4y \overline{\Psi_S} \Xi \quad (16)$$

where $\Xi = \xi(x, \tau, j, \alpha) \otimes \zeta(y, \tau, k, \beta)$ is a simple tensor product of wavefunctions for two rotation eigenstates respectively having eigenvalues j, k and frame angles α, β . It is readily deduced from (7) that f is independent of τ . In particular the ‘*to*’ state Ψ is symmetrized in the sense of (15), but not the ‘*from*’ state Ξ . If $l = n$ then f is effectively independent of all the frame angles $\alpha, \beta, \chi, \lambda$. The orthogonality of the rotation eigenstates implies that, in the case $l = -n$, the transition amplitude f vanishes if $j = k$. If $l = -n$ and $j = -k$, then f reduces to a single term, that is, there is no interference between two summands. Furthermore f is then proportional either to unity or to κ , and so the physically significant $|f|^2$ is independent of the frame angles. The preceding argument would fail if the *from* state Ξ were also symmetrized as in (15), for then the summands in f would not reduce in the case of different rotations to a single nonvanishing term which is effectively independent of the

frame angles. To continue the construction, the factor κ in (15) may therefore and without loss of generality be set to unity even if $l \neq n$, yielding the antisymmetric wavefunction

$$\Psi_S = \psi(x, \tau, l, \chi) \otimes \phi(y, \tau, n, \lambda) - \phi(x, \tau, n, \lambda) \otimes \psi(y, \tau, l, \chi). \quad (17)$$

Jabs further points out that, if and only if $\kappa = 1$, the Feynman transition amplitude f in (16) coincides with the standard transition amplitude g defined by

$$g = \int d^4x \int d^4y \frac{1}{\sqrt{2}} \bar{\Psi}_S \frac{1}{\sqrt{2}} \Xi_S. \quad (18)$$

Both the *from* and *to* states are antisymmetrized in g , and both are normalized with factors of $1/\sqrt{2}$.

3. summary of principles

The antisymmetry of the two-particle wavefunction for indistinguishable spin-1/2 particles, each of which is in either of the two eigenstates of the same rotation operator $R(s)$ where $s = (0, \mathbf{s})$ and $s \cdot s = 1$, follows [7] from three principles.

- P1 The two-particle wavefunction resulting from an exchange of two particles, both in eigenstates of the same rotation operator $R(s)$, must be added to the original two-particle wavefunction for the purpose of calculating transition amplitudes.
- P2 The exchange of the two unspecified frame-dependent angles must be homotopically consistent, in the case of same rotation ($l = n$) resulting in an antisymmetric two-particle wavefunction as in (12) or (17). In the case of different rotations ($l = -n$) the resulting factor $-\kappa$ as in (15) may be replaced with negative unity, also yielding an antisymmetric two-particle wavefunction as in (17)
- P3 Transition amplitudes must be calculated as in (16) using (17), that is, after antisymmetrizing the *to* state but not the *from* state. The result agrees with the standard amplitude (18), for which both two-particle states are exchanged and normalized with factors of $1/\sqrt{2}$.

Again, the addition principle in P1 and the transition amplitude in P3 owe to Feynman [24].

B. At $\tau > 0$

Consider two free and indistinguishable spin-1/2 particles in plane wave states, both of which are for example positive-energy and propagating forward in time at $\tau > 0$. Their wavefunctions are not necessarily eigenvectors of any rotation operators. It may be assumed that for any one pure spacelike four-vector s the two-particle wavefunction at parameter $\tau > 0$ evolves from a pair of free wavefunctions for states prepared at $\tau = 0$, both of which wavefunctions are eigenvectors of the same rotation operator $R(s)$. The two-particle wavefunction at $\tau = 0$ must therefore be antisymmetric. The separable free influence function (8) preserves antisymmetry, and so the two-particle wavefunction must be antisymmetric for any $\tau > 0$. The two-particle influence function is also [14] separable in the presence of an electromagnetic field $A^\mu(x)$, and so antisymmetry is preserved even if there is a field in the space-time neighborhoods of the particles at $\tau > 0$. If the particles are not in free states at $\tau > 0$, then provided they evolve from states which are free at some $\rho > 0$ where $\tau > \rho > 0$, it may be concluded that the two-particle wavefunction is antisymmetric or fermionic at τ .

C. Other Spins

No fundamental particles of half-integer spin greater than 1/2 have been observed. Excluding the gauge bosons (which may be treated semiclassically as the photons are here), the only fundamental integer-spin particle that has been observed is the Higgs boson of spin-0 [25, 26]. There are of course mesons of spin-0 and spin-1. Cooper pairs are typically in the singlet state [27], that is, as spin-0 bosons for which the relativistic wave equation is the Klein-Gordon equation [1, 6]. Principle P1 suffices for spin-0, since there are no rotation frame angles and the transition amplitudes f and g are the same. Cooper pairs have been observed in the triplet state [27], that is, as spin-1 bosons for which there is the Duffin-Kemmer-Petiau or DKP relativistic wave equation [9–11]. The parametrized DKP equation has the same form as the parametrized Dirac equation (1), with the matrices γ^μ obeying the fermion algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2g^{\mu\nu} \quad (19)$$

replaced by the matrices β^μ obeying the meson algebra

$$\beta^\lambda \beta^\mu \beta^\nu + \beta^\nu \beta^\mu \beta^\lambda = -\beta^\lambda g^{\mu\nu} - \beta^\nu g^{\mu\lambda}. \quad (20)$$

The argument for spin-1 closely parallels that presented above for the parametrized Dirac equation, but arrives instead at symmetric wavefunctions.

V. DISCUSSION

A. Causality

The standard proof of the spin-statistics connection in QFT requires the causality conditions that are incorporated into the commutators and anticommutators for bosonic and fermionic fields respectively [2]. The causality conditions are defined by the light cone in x with apex at y , where x and y are the two events (the arguments of the two fields in the operator products). The causality conditions in the commutators and anticommutators are also required for \mathcal{TPC} invariance [28], and for the covariance of Dyson series [2]. No causality condition is required here for the proof of the spin-statistic connection. Furthermore, \mathcal{TPC} invariance is an immediate consequence of (1), while Dyson series for (1) are manifestly covariant since they are τ -ordered rather than t -ordered.

Applications of the parametrized Dirac equation to interactions involve semiclassical electromagnetic fields, and the fields are constructed from Møller currents using the standard Feynman propagator which does impose a causality condition [6, 14]. The single-fermion influence function Γ_+^0 in (6) allows free wavefunctions only of positive mass[29] to evolve for τ increasing, and only those of negative mass[30] for τ decreasing. The detailed form of $\Gamma_+^0(x - x', \tau - \tau')$ does involve the light cone in x with apex at x' , although propagation is not sharp, but the two-fermion influence function $\Gamma_{++}^0(x - x', y - y', \tau - \tau')$ is separable in x and y .

B. Entanglement

The first-quantized formalism considered here is entirely different from QFT. Entanglement is immediately implied by the simple antisymmetric wavefunction in (12), and is conserved with respect to τ by (7), but is a more complex concept in a quantum field [31]. Some remarks on the nature of entanglement in the two formalisms are therefore in order.

The parametrized Dirac formalism is not unitary and so an invariant probability density is not available for the definition of entanglement. The antisymmetric wavefunction Ψ_S

in (12) expresses the entanglement of two particles for each τ in the following sense: the covariant current [14] associated with Ψ_S is not the sum of the individual currents for the particles. Of all linear combinations of the simple tensor products in (12), the antisymmetric Ψ_S yields the maximal nonadditivity (maximal interference). The entanglement in Ψ_S is not restricted by $x - y$, but the current nonadditivity at x is negligible if the two single-particle wavefunctions have negligible overlap as in $\int d^4y \bar{\psi}\phi \sim 0$.

The entanglement of a quantum field itself is expressed in terms of correlation. For a massive vacuum field, the exponential decay scale for correlations with space-like separation is the Compton wavelength [31]. The proof [32–34] assumes that the field operators at x and y (after propagating to a common coordinate time t) commute for space-like $x - y$, that is, a causality condition is assumed. It has been argued [35] that the decay scale of entanglement for two qubits, which have space-like separation L and which are interacting with a scalar vacuum during a time $T \ll L/c$, is cT . Entanglement for the two qubits is defined as the nonseparability of a density matrix [36]. For times less than L/c , the state of one qubit is independent of the other even though there is a correlation in the field [37]. Tests of these remarkable findings have been proposed using circuit QED [38] and cavity QED [39]. The situation here is analogous. The semiclassical electromagnetic field arising from the entangled covariant current is subject to a causality condition, as a consequence of specifying the Feynman propagator. Thus a classical detector will at first receive only the signal from the closer of two entangled particles if the current nonadditivity is negligible. Spin-entangled spatially separated electron pairs can now be produced by splitting Cooper pairs, with confirmation of their entanglement approaching feasibility [40].

C. Summary

Particles of spin-1/2 prepared at the parameter value $\tau = 0$ need not satisfy the standard Dirac equation for a sharp mass, even though the wave packets may be extensive in all four dimensions of space-time at $\tau = 0$. The particles may therefore all be prepared in eigenstates of the rotation operator $R(s)$ for a single pure spacelike rotation axis s^μ and so, if the particles are indistinguishable and if exchanges of states include homotopically-consistent exchanges of the rotation frame angles, Fermi-Dirac statistics are inferred. Subsequent evolution for $\tau > 0$ preserves such statistics, even though each particle may no longer be in a rotation

eigenstate. The relativistic spin-statistics connection obtains here so naturally because the parametrized Dirac equation is manifestly covariant as an evolution equation.

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APPENDIX: LINEAR ALGEBRA

Given $w_+ = \mathbf{w}_+ \mathbf{a}_+ = \Lambda_+ w_+$, we seek w_- such that $w_- = \Lambda_- w_-$ and $R(s)w = +(1/2)w$ where $s = (0, \mathbf{s})$ and $w = w_+ + w_-$. The required w_- must satisfy

$$Q(s)\Lambda_- w_- = -Q(s)w_+, \quad (21)$$

which has a solution if and only if $\bar{z}Q(s)w_+ = 0$ for all z such that $\bar{z}Q(s)\Lambda_- = 0$. It suffices to consider a frame in which $s^\mu = (0, 0, 0, 1)$, and so $Q(s) = \text{diag}(0, 1, 0, 1)$. Assuming

$\mathbf{p} \times \mathbf{s} = (p_2, -p_1, 0) \neq \mathbf{0}$, it follows that $z_2 = z_4 = 0$ while z_1 and z_3 are arbitrary. Hence $\bar{z}Q(s) = 0$. The solution of (21) for w_- is undetermined up to the addition of $\Lambda_+ b$ for any b , but $w_- = \Lambda_- w_-$ is uniquely determined.

Consider two free particles with timelike energy-momenta p^μ and q^μ , where $\mathbf{p} \times \mathbf{s} = (p_2, -p_1, 0) \neq \mathbf{0}$ but $\mathbf{q} \times \mathbf{s} = (q_2, -q_1, 0) = \mathbf{0}$. There is a boost Ω to a new frame where $p' = \Omega p$ and $q' = \Omega q$, with $(p'_1, p'_2) \neq (0, 0)$ and $(q'_1, q'_2) \neq (0, 0)$. The common rotation axis $(0, 0, 0, 1)$ in the old frame may be replaced with $(0, 0, 0, 1)$ in the new frame. The procedure may be performed any finite number of times for any finite number of particles, with $(p'_1, p'_2), (q'_1, q'_2), \dots$ all remaining bounded away from $(0, 0)$.